INTERSECTIONS BY HYPERPLANES*

BY

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ABSTRACT

Let $\mathscr{A}$ be a finite nonempty family of nonempty disjoint closed and bounded sets in a Banach space E which is either separable and the conjugate of some Banach space X (i.e. $E = X^*$) or, reflexive and locally uniformly convex. If C denotes the weak*-closed convex hull of $\cup\{A: A \in \mathscr{A}\}$ then the set of points in $E \sim C$ through which there is no hyperplane intersecting exactly one member of $\mathscr{A}$ is discrete (or empty).

1. Introduction. In a recent paper by L. M. Kelly [3] an example credited to V. Klee, is given showing that in c_0 two disjoint closed and bounded sets exist such that every closed hyperplane in that space either intersects both or none. That for compact subsets of any real locally convex Hausdorff space this cannot occur is shown in [2]. Indeed we have proved there that for any finite nonempty family $\mathscr{A}$ of nonempty disjoint compact sets in such a space X there is always a closed hyperplane intersecting exactly one member of $\mathscr{A}$. Moreover, if $B = \cup\{A: A \in \mathscr{A}\}$ is not a subset of a straight line then the set of points in $X \sim \overline{\text{co}B}$ through which there is no hyperplane intersecting a single member of $\mathscr{A}$ is discrete. It is the purpose of the present note to prove that the last assertion remains true for families of closed bounded sets in a certain class of Banach spaces. The class in question contains all separable conjugate spaces and all those which are reflexive and locally uniformly convex.

2. Lemma 1. *Let X^* be the conjugate space of a Banach space X, A a nonempty closed and bounded set in X^*, B the weak*-closed convex hull of A, x any point in $X^* \sim B$ and C the cone spanned by x and B. Let π be a weak*-closed hyperplane strictly separating x and B and set $K = C \cap \pi$. If e is any extreme point of K at which the identity mapping $i:(K, \text{weak*}) \to (K, \text{norm})$ is continuous and R is the ray through e (and x) then the extreme points of $R \cap B$ belong to A.*

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Proof. We observe that the extreme points of $R \cap B$ are extreme points of the weak*-compact set B and by a known theorem of Milman (cf. [4, p. 335]) they must belong to the weak*-closure of A. Let r be an extreme point of $R \cap B$ and suppose $r \notin A$. To prove the lemma it suffices to show that then r cannot be in the weak*-closure of A. Now, either $[x,r]$ or $R \sim [x,r)$ (or both) are disjoint from A. The proofs in both cases being, with minor changes, similar we deal here with the first only. Let then $r' \in R$ be such that $[x, r] \subset [x, r')$ and $[x,r'] \cap A \neq \emptyset$ and set $\lambda = (\| x - r' \|)/(\| x - e \|)$. Let D be an open ball around the origin such that $[x,r'] \cap A = \emptyset$. By the continuity of i at e there is a weak*-open neighborhood W of the origin such that $(\lambda W + e) \cap K \subset (D + e) \cap K$. Let V be the cone spanned by x and $(W + e) \cap K$. Then $V \cap ([x,r'] + D)$ is readily seen to be a weak*-neighborhood of r in B which is disjoint from A, so that the weak*-closure of A fails to contain r.

LEMMA 2. *Let X^* be the conjugate space of a Banach space X and $\mathscr{A}$ a nonempty finite family of nonempty disjointed closed and bounded sets in X^*, let $G = \cup \{A : A \in \mathscr{A}\}$ and suppose B is the weak*-closed convex hull of G, C is the cone spanned by B and by any $x \in X^* \sim B$. Suppose further that R is an extreme ray of C with the property that the extreme points of $R \cap B$ belong to G.*

Then there is a weak-open neighborhood W of x such that for any $p \in W \sim \bar{R}$, where $\bar{R}$ is the straight line containing R, there is at least one extreme point q of $R \cap B$ such that the straight line L through p and q intersects exactly one member of $\mathscr{A}$ and is disjoint from the weak*-closed convex hull of the union of the remaining members of $\mathscr{A}$.*

Proof. Suppose, first, that $R \cap B$ is a singleton r in which case there is a unique A_r such that $r \in A_r \in \mathscr{A}$. Let $G_r = \cup \{A : A \in \mathscr{A},\ A \neq A_r\}$ and let B_r be the weak*-closed convex hull of G_r. Since $r \notin G_r$ it follows from Lemma 1 that $B_r \cap \bar{R} = \emptyset$. Moreover, if C_r is the cone spanned by r and B_r then x is not in C_r nor in $2r - C_r$, the set symmetric to C_r with respect to r. The weak*-open set $X^* \sim (C_r \cup (2r - C_r))$ is now easily seen to be a neighborhood W as desired.

The same argument is applicable to the more general case when R is disjoint from all but one $A_r \in \mathscr{A}$ (even though $R \cap B$ may not be a singleton).

Suppose now $R \cap B$ is not a singleton and R intersects at least two members of $\mathscr{A}$. Let r and s be the two extree points of $R \cap M$ with $r \in (x,s)$ and suppose $r \in A_r \in \mathscr{A}$, $s \in A_s \in \mathscr{A}$ (where A_r and A_s are not necessarily distinct), let G_r and B_r

be defined as in the first case with G_s and B_s similarly defined for s. Here $[x,r] \cap B_r = \emptyset$ and $(R \sim [x,s)) \cap B_s = \emptyset$ so that weak*-closed hyperplanes λ and μ exist the first strictly separating $[x,r]$ and B_r, the second $R \sim [x,s)$ and B_s. We denote by λ^+, μ^+ and π^+ the open halfspaces determined by λ, μ and π respectively and containing x. If now $p \in \lambda^+ \cap \mu^+ \cap \pi^+ \sim \bar{R}$ then at least one of the straight lines joining p with r and s may serve as the L of this lemma. (Indeed in the plane through w, r and s the points of intersection, if any, of these lines with B_r and B_s respectively must lie on opposite sides of $\bar{R}$; since R is an extreme ray at least one of these must be empty.)

LEMMA 3. *Let E be a Banach space satisfying at least one of the following two conditions*:

(i) *It is the conjugate of some Banach space X* (i.e. $E = X^*$) *and separable*;

(ii) *It is reflexive and locally uniformly convex (but not necessarily separable).*
Let A be a closed and bounded set in E containing at least two points, K the weak-closed convex hull of A. Then K contains at least two extreme points at which the identity mapping* $i:(K,\text{weak*}) \to (K,\text{norm})$ *is continuous.*

Proof. In spaces satisfying condition (i) the assertion is a simple consequence of a deep result of I. Namioka [6] according to which the set of all extreme points of K at which i is continuous is weak*-dense in $\operatorname{ext} K$, the set of all extreme points of K. Indeed, if π is a weak*-closed hyperplane strictly separating two points of A then each of the open halfspaces determined by π must contain at least one extreme point of K (for K is the weak*-closed convex hull of $\operatorname{ext} K$); hence also a point of $\operatorname{ext} K$ at which i is continuous.

For spaces satisfying (ii) the weak*-topology is identical with the weak topology and, as pointed out by J. Lindenstrauss each point on the boundary of the unit ball $U = \{x: \|x\| \leq 1\}$ is strongly exposed (cf. [5, p. 140]) and a *fortiori* i is continuous there. Our assertion now follows immediately from a theorem of E. Asplund [1] according to which the set of all points in E, (where E satisfies (ii)), having a farthest point in A is dense in E. For if A contains at least two points then it clearly also must contain at least two points each of which is farthest from some point in E; and since K is contained in the intersection of all closed balls containing A it follows that i is continuous at these points.

REMARK. It is easily seen that points of K at which i is continuous belong to A (cf. also [6], Theorem 3.5).

3. THEOREM. *Let E be a Banach space which is either the conjugate of some Banach space and separable, or reflexive and locally uniformly convex. Let $\mathscr{A}$ be a finite nonempty family of nonempty disjoint closed and bounded sets in E and suppose that $G = \cup \{A : A \in \mathscr{A}\}$ is not a subset of any straight line of E. Then the set of all points in $E \sim B$, where B is the weak*-closed convex hull of G, through which there is no weak*-closed hyperplane intersecting one member of $\mathscr{A}$ is discrete.*

Proof. Let $x \in E \sim B$ and suppose C is the cone spanned by x and C. Since B is weak*-compact there exists a weak*-closed hyperplane strictly separating x and B. The set $K = C \cap \pi$ is convex and weak*-compact [4, p. 341] and, by Lemma 3, contains at least two extreme points at which the identity mapping $i:(K, \text{weak*}) \to (K, \text{norm})$ is continuous. Hence there are two rays R_1 and R_2 for each of which the conclusion of Lemma 1 is true. Lemma 2 applies to each of these rays and it follows that weak*-open neighborhoods W_i, $i = 1,2$ of x exist satisfying the conclusion of that lemma with respect to R_i. Clearly, then, for each $p \in W_1 \cap W_2$ there is a point r either on $R_1 \cap B$ or on $R_2 \cap B$ such that the line L through p and r intersects exactly one A_r with $r \in A_r \in \mathscr{A}$ and L is disjoint from B_r, the weak*-closed convex hull of $\cup \{A : A \in \mathscr{A}, A \neq A_r\}$. It follows that a weak*-closed hyperplane λ exists which strictly separates L and B_r. But then the hyperplane parallel to λ and containing L goes through x and is disjoint from all other members of $\mathscr{A}$.

4. REMARKS.

Since the theorem remains true for any isomorphic image of E the property of local uniform convexity can clearly be replaced by that of having a norm which is equivalent to a locally uniformly convex one.

In a recent paper, E. Asplund* introduced the notion of a strong differentiability space (SDS), defined as a Banach space with the property that every convex function $f : X \to (-\infty, \infty]$ is Fréchet differentiable on a dense G_δ subset of its domain of continuity, and showed that the spaces referred to in Lemma 3 (or those with an equivalent norm) are conjugates of an SDS space. On the other hand, for a space E which is the conjugate of an SDS the truth of our Lemma 3 follows from Asplund's Proposition 5 and the remark at the end of its proof** .Thus our theorem is seen to be true in the more general setting of a space E which is the conjugate of an SDS.

* *Acta* Math. 121 (1968), 31–47.
** *Ibid.*, pp. 46–47.

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